of Solid Bodies and Fluids. Investigation by the Neutron Scattering Method, Mir, Moscow, 1980.
6. SEDOV L.I., Mechanics of a Continuous Medium., 1, Nauka, Moscow, 1973.
7. HORMANDER L., Analysis of linear partial differential operators. 1. Theory of Distributions and Fourier Analysis, Mir, Moscow, 1986.

Translated by M.D.F.

PMM U.S.S.R.,Vol.54,No.1,pp.51-59,1990
Printed in Great Britain

0021-8928/90 $10.00+0.00
©1991 Pergamon Press plc

FIRST AND SECOND FUNDAMENTAL AXISYMMETRIC PROBLEMS OF ELASTICITY THEORY FOR DOUBLY-CONNECTED DOMAINS BOUNDED BY THE SURFACES OF A SPHERE AND A SPHEROID*

A.G. NIKOLAYEV and V.S. PROTSENKO

The method of constructing solutions of the fundamental boundary-value problems for a homogeneous Lamé equation of multiconnected domains bounded by canonical surfaces of cylindrical and spheroidal coordinate systems described in /1/ is extended to domains with other geometry. The problems under consideration reduce to infinite systems of linear algebraic equations of the second kind with completely continuous opertors. A solution in the form of expansions in a small parameter for the problem of the hydrostatic pressure of a sphere with a centrally located spheroidal cavity is presented as an example.

1. We consider the first and second fundamental axisymmetric problems for a homogeneous Lamé equation

$$\nabla^2 \mathbf{u} + (1-2\nu)^{-1}\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} = 0 \tag{1.1}$$

(ν is Poisson's ratio) for a sphere with a spheroidal cavity whose axis passes through the centre of the sphere. Introducing identically directed systems of spherical coordinates (r, θ, φ) and prolate spheroidal coordinates (ξ_1, η_1, φ) superposed on the centres of the boundary surfaces, we obtain the following relation between the coordinates

$$r\cos\theta = c\,\mathrm{ch}\,\xi_1\cos\eta_1 + a,\quad r\sin\theta = c\,\mathrm{sh}\,\xi_1\sin\eta_1 \tag{1.2}$$

($2c$ is the focal length of the spheroidal system of coordinates, and a is the spacing between the centres of the boundary surfaces).

Let displacement vectors be given on the boundary

$$\mathbf{U}_{|r=R} = \sum_{k=0}^{\infty}\left[A_{k,1}^{(1)}P_k^{(1)}(\cos\theta)\,\mathbf{e}_\rho + A_{k,1}^{(2)}P_k(\cos\theta)\,\mathbf{e}_z\right] \tag{1.3}$$

$$\mathbf{U}_{|\xi_1=\xi_0} = \sum_{k=0}^{\infty}\left[A_{k,2}^{(1)}P_k^{(1)}(\cos\eta_1)\,\mathbf{e}_\rho + A_{k,2}^{(2)}P_k(\cos\eta_1)\,\mathbf{e}_z\right]$$

(e_ρ, e_z are unit vectors of the cylindrical system of coordinates). We later assume that

$$\sum_{k=0}^{\infty} k\left(k\,|A_{k,i}^{(1)}| + |A_{k,i}^{(2)}|\right) < \infty \quad (i=1,2) \tag{1.4}$$

We will seek the solution of problem (1.1) and (1.3) in the form

$$\mathbf{U} = \sum_{k=1}^{2}\sum_{n=0}^{\infty}\left[a_n^{(k)}\frac{n!}{R^n}\mathbf{W}_{k,n}^-(r,\theta) + \frac{a_n^{(k+2)}}{Q_n(\mathrm{ch}\,\xi_0)}\mathbf{U}_{k,n}^+(\xi_1,\eta_1)\right] \tag{1.5}$$

*Prikl.Matem.Mekhan.,54,1,65-74,1990

The following notation is used here and henceforth:

$$\mathbf{W}^{\pm}_{k,n} = \mathbf{D}^{(1)}_{k}[w^{\pm}_{k,n}(r,\theta,\varphi)],\quad \mathbf{U}^{\pm}_{k,n} = \mathbf{D}^{(2)}_{k}[u^{\pm}_{k,n}(\xi_1,\eta_1,\varphi)]\quad (k=1,2) \tag{1.6}$$
$$\mathbf{D}^{(1)}_1 = \operatorname{grad},\quad \mathbf{D}_2 = z\operatorname{grad} + (4\nu-3)\mathbf{e}_z,\quad D^{(2)}_1 = \frac{c}{2n+1}\mathbf{D}^{(1)}_1$$
$$\mathbf{D}^{(1)}_2 = \mathbf{D}_2 - R^2\mathbf{D}^{(1)}_1 D^{\pm}_1,\quad \mathbf{D}^{(2)}_2 = \mathbf{D}_2 - c\operatorname{ch}^2\xi_0\mathbf{D}^{(1)}_1 D^{\pm}_2$$

$$D^{\pm}_2 u^{\pm}_n = u^{\pm}_{n\pm1},\ D^{\pm}_1 w^{\pm}_n = [2(n\pm1)+1]^{-1}w^{\pm}_{n\pm1},\quad w^{\pm}_{1,n} = w^{\pm}_{n\mp1}$$
$$w^{\pm}_{2,n} = w^{\pm}_n,\quad u^{\pm}_{1,n} = (D_2^- - D_2^+)u^{\pm}_n,\quad u^{\pm}_{2,n} = u_n^{\pm}$$
$$w_n^{\pm} = \begin{Bmatrix}\dfrac{n!}{r^{n+1}}\\[2mm] \dfrac{r^n}{n!}\end{Bmatrix}P_n(\cos\theta),\quad u_n^{\pm} = \begin{Bmatrix}Q_n(\operatorname{ch}\xi)\\ P_n(\operatorname{ch}\xi)\end{Bmatrix}P_n(\cos\eta),\quad \begin{matrix}q=\operatorname{ch}\xi_0\\ \bar q = \operatorname{sh}\xi_0\end{matrix}$$

where $P_n^{(m)}(x)$ and $Q_n^{(m)}(x)$ are Legendre function of the first and second kind.

The solution (1.6) are an axisymmetric modification of the exact solutions for a sphere and spheroid introduced in /2/. The exact solutions for a sphere and spheroid were examined in somewhat different form in /3, 4/.

We will change to spherical coordinates in the general solution (1.5) by using the formulas for the expansion of spherical solutions of the Lamé equation in spheroidal solutions /2/

$$\mathbf{W}^{-}_{1,n} = \sum_{k=0}^{n}(-1)^{k+n+1}\left(k+\frac{1}{2}\right)C^{(1)}_{n,k}\mathbf{U}^{-}_{1,k} \tag{1.7}$$
$$\mathbf{W}^{-}_{2,n} = \sum_{k=0}^{n}(-1)^{k+n}\left(k+\frac{1}{2}\right)[C^{(1)}_{n,k}\mathbf{U}^{-}_{2,k} + C^{(2)}_{n,k}\mathbf{U}^{-}_{1,k}]$$
$$C^{(1)}_{n,k} = \sqrt{\pi}\left(\frac{c}{2}\right)^n\frac{1}{\Gamma\left(n+\frac{3}{2}\right)}C_{n-k}^{-\frac{1}{2}-n}\left(\frac{a}{c}\right),$$
$$C^{(2)}_{n,k} = \frac{R^2}{2n-1}C^{(1)}_{n-2,k} - cq^2C^{(1)}_{n-1,k+1}$$

($C_m^{\nu}(x)$ is the Gegenbauer function). Substituting (1.7) into (1.5), we obtain after some reduction

$$\mathbf{U} = \sum_{k=0}^{\infty}\left[\frac{a^{(3)}_k}{Q_k(q)}\mathbf{U}^{+}_{1,k} + \frac{a^{(4)}_k}{Q_k(q)}\mathbf{U}^{+}_{2,k} + \left(k+\frac{1}{2}\right)\sum_{i=1}^{2}H^{(1)}_{ik}\mathbf{U}^{-}_{i,k}\right] \tag{1.8}$$
$$H^{(1)}_{ik} = \sum_{n=k}^{\infty}a^{(i)}_n\frac{n!}{R^n}(-1)^{i+k+n}C^{(1)}_{n,k} + \delta_{i1}\sum_{n=k}^{\infty}a^{(2)}_n\frac{n!}{R^n}(-1)^{k+n}C^{(2)}_{n,k},$$
$$\delta_{i1} = \begin{cases}1, & i=1\\ 0, & i\neq 1\end{cases}$$

We similarly convert (1.5) to spherical coordinates by using formulas for the expansion of spheroidal solutions of the Lamé equation in spherical solutions /2/

$$\mathbf{U}^{+}_{1,n} = \frac{c}{2}\sum_{k=n}^{\infty}(-1)^{k+n}C^{(1)}_{k,n}\mathbf{W}^{+}_{1,k} \tag{1.9}$$
$$\mathbf{U}^{+}_{2,n} = \frac{c}{2}\sum_{k=n}^{\infty}(-1)^{k+n}[C^{(1)}_{k,n}\mathbf{W}^{+}_{2,k} + C^{(2)}_{k,n}\mathbf{W}^{+}_{1,k}]$$

We consequently have

$$\mathbf{U} = \sum_{k=0}^{\infty}\left[a^{(1)}_k\frac{k!}{R^k}\mathbf{W}^{-}_{1,k} + a^{(2)}_k\frac{k!}{R^k}W^{-}_{2,k} + \frac{c}{2}\sum_{i=1}^{2}H^{(2)}_{ik}W^{+}_{i,k}\right] \tag{1.10}$$
$$H^{(2)}_{ik} = \sum_{n=0}^{k}\frac{a^{(i+2)}_n(-1)^{k+n}}{Q_n(q)}C^{(1)}_{k,n} + \delta_{i1}\sum_{n=0}^{k}\frac{a^{(4)}_n(-1)^{k+n}}{Q_n(q)}C^{(2)}_{k,n}$$

Changing to the coordinate mode of writing the displacements (1.8) and (1.10) in the basis e_ϱ, e_z and satisfying the conditions on the boundary (1.3) by using the orthogonality of the Legendre functions we obtain an infinite system of linear algebraic equation in $a_k^{(p)}$

$$\sum_{p=1}^{2}(s_{k,p}^{(i)}a_k^{(p)}+\sum_{n=0}^{k}a_n^{(p+2)}t_{n,k}^{(i,p)})=A_{k,1}^{(i)} \tag{1.11}$$

$$\sum_{p=1}^{2}\left(s_{k,p}^{(i+2)}a_k^{(p+2)}+\sum_{n=k}^{\infty}a_n^{(p)}t_{n,k}^{(i+2,p)}\right)=A_{k,2}^{(i)}$$

$$(i=1,2;\ p=1,2,3,4;\ k=1,2,\ldots)$$

Here

$$s_{k,1}^{(1)}=\frac{1}{k+1},\quad s_{k,2}^{(1)}=\frac{k-1}{2k-1},\quad s_{k,1}^{(2)}=1,\quad s_{k,2}^{(2)}=4\nu-3+\frac{k^2}{2k-1} \tag{1.12}$$

$$s_{k,1}^{(3)}=-\frac{Q_k^{(-1)}(q)}{Q_k(q)},\quad s_{k,2}^{(3)}=-\frac{(k+2)\,qQ_k^{(-1)}(q)}{Q_k(q)}$$

$$s_{k,1}^{(4)}=-1,\quad s_{k,2}^{(4)}=\frac{(4\nu-3)\,Q_k(q)-(k+1)\,qQ_{k+1}(q)}{Q_k(q)}$$

$$t_{n,k}^{(1,1)}=\frac{(-1)^{k+n}c\,(k-1)!}{2R^{k+1}Q_n(q)}C_{k,n}^{(1)},\quad t_{n,k}^{(2,1)}=\frac{(-1)^{k+n+1}ck!}{2R^{k+1}Q_n(q)}C_{k,n}^{(1)}$$

$$t_{n,k}^{(1,2)}=\frac{(-1)^{k+n}c\,(k-1)!}{2R^{k+1}Q_n(q)}\left[C_{k,n}^{(2)}+\frac{k\,(k+2)}{2k+3}C_{k,n}^{(1)}\right]$$

$$t_{n,k}^{(2,2)}=\frac{(-1)^{k+n}ck!}{2R^{k+1}Q_n(q)}\left[\left(4\nu-3-\frac{(k+1)^2}{2k+3}\right)C_{k,n}^{(1)}-C_{k,n}^{(2)}\right]$$

$$t_{n,k}^{(3,1)}=(-1)^{k+n}\left(k+\frac{1}{2}\right)\frac{n!}{R^n}P_k^{(-1)}(q)\,C_{n,k}^{(1)}$$

$$t_{n,k}^{(4,1)}=(-1)^{k+n}\left(k+\frac{1}{2}\right)\frac{n!}{R^n}P_k(q)\,C_{n,k}^{(1)}$$

$$t_{n,k}^{(3,2)}=(-1)^{k+n}\left(k+\frac{1}{2}\right)\frac{n!}{R^n}\left[(k-1)\,qP_{k-1}^{(-1)}(q)\,C_{n,k}^{(1)}-P_k^{(-1)}(q)\,C_{n,k}^{(2)}\right]$$

$$t_{n,k}^{(4,2)}=(-1)^{k+n}\left(k+\frac{1}{2}\right)\frac{n!}{R^n}\left[((4\nu-3)\,P_k(q)+kqP_{k-1}(q))\,C_{n,k}^{(1)}-P_k(q)\,C_{n,k}^{(2)}\right]$$

For $k=0$ it is necessary to append $a_0^{(1)}=a_0^{(3)}=0$ to (1.11) for $i=2$. Solving (1.11) for $a_k^{(p)}$, we represent the infinite system in the form

$$a_k^{(i)}+(-1)^{i+1}\sum_{p=1}^{2}\sum_{n=0}^{k}a_n^{(p+2)}\,\frac{t_{n,k}^{(1,p)}s_{k,3-i}^{(2)}-t_{n,k}^{(2,p)}s_{k,3-i}^{(1)}}{\Delta_1}= \tag{1.13}$$

$$(-1)^{i+1}\,\frac{A_{k,1}^{(1)}s_{k,3-i}^{(2)}-A_{k,1}^{(2)}s_{k,3-i}^{(1)}}{\Delta_1},\quad \Delta_1=s_{k,1}^{(1)}s_{k,2}^{(2)}-s_{k,1}^{(2)}s_{k,2}^{(1)}$$

$$a_k^{(i+2)}+(-1)^{i+1}\sum_{p=1}^{2}\sum_{n=k}^{\infty}a_n^{(p)}\,\frac{t_{n,k}^{(3,p)}s_{k,3-i}^{(4)}-t_{n,k}^{(4,p)}s_{k,3-i}^{(3)}}{\Delta_2}=$$

$$(-1)^{i+1}\,\frac{A_{k,2}^{(1)}s_{k,3-i}^{(4)}-A_{k,2}^{(2)}s_{k,3-i}^{(3)}}{\Delta_2},\quad \Delta_2=s_{k,1}^{(3)}s_{k,2}^{(4)}-s_{k,2}^{(3)}s_{k,1}^{(4)}$$

For $k=0$ it is necessary to append $a_0^{(1)}=a_0^{(3)}=0$ to the equations

$$a_0^{(2)}+\frac{t_{0,0}^{(2,2)}}{s_{0,2}^{(2)}}\,a_0^{(4)}=\frac{A_{0,1}^{(2)}}{s_{0,2}^{(2)}},\quad a_0^{(4)}=\sum_{p=1}^{2}\sum_{n=0}^{\infty}a_n^{(p)}\,\frac{t_{n,0}^{(4,p)}}{s_{0,2}^{(4)}}=\frac{A_{0,2}^{(2)}}{s_{0,2}^{(4)}} \tag{1.14}$$

Lemma 1.1. For $k\geqslant 1,\ \nu<{}^1/_2$ the determinants Δ_1, Δ_2 are non-zero for all $\xi_0>0$. The following estimates hold:

$$|\Delta_1|\geqslant(2-4\nu)\,(k+1)^{-1},\quad |\Delta_2|\geqslant(3-4\nu)\,(k+1)^{-1} \tag{1.15}$$

Proof. On the second estimate is needed in the proof. Writing Δ_2 in explicit form

$$Q_k(q)^2\,\Delta_2=\frac{q}{k}\,Q_k^{(1)}(q)\,Q_{k+1}(q)-\frac{q}{k+1}\,Q_k(q)\,Q_{k+1}^{(1)}(q)+\frac{3-4\nu}{k\,(k+1)}\,Q_k(q)\,Q_k^{(1)}(q)$$

we note that the last component is negative. Using the integral representation of the Legenre function of the second kind, a formula

$$\frac{q}{k}Q_k^{(1)}(q)\,Q_{k+1}(q)-\frac{q}{k+1}Q_k(q)\,Q_{k+1}^{(1)}(q)=-q\bar{q}^2\int_0^\infty dt\int_0^t\frac{(\operatorname{ch}t-\operatorname{ch}u)^2\,du}{(q+\bar{q}\operatorname{ch}t)^{k+2}(q+\bar{q}\operatorname{ch}u)^{k+2}}$$

can be obtained for the first two components from which and from the inequality $|Q_k^{(1)}(q)|>kQ_k(q)$ the required estimate follows.

The infinite system (1.13) and (1.14) is the operator equation $(I+T)x=f$, where x, f are columns of the unknowns and the right-hand sides, respectively, I is the unit operator, and T is the system operator.

Lemma 1.2. The operator T of system (1.13) and (1.14) is a completely continuous operator from l_2 into l_2 for $R>c\operatorname{ch}\xi_0+a$.

Proof. As we know /5/, it is sufficient to show for the proof that the matrix coefficients of the operator are square summable. It follows directly from relations (1.12) that an arbitrary matrix coefficient can have the sum of a finite number of expressions of the form

$$\tau_{n,k}=Bn^\alpha P_n^{(\gamma)}(q)\,(k+\beta)!\,R^{-k}\,|C_{k,n}^{(1)}|$$

as the upper bound, where B is a certain constant independent of n and k and α,β,γ are fixed non-negative integers. We note that

$$\sum_{n=0}^\infty\sum_{k=0}^\infty|\tau_{n,k}|\leqslant B_1\sum_{n=0}^\infty n^\alpha P_n^{(\gamma)}(q)\,Q_n^{(\beta)}\left(\frac{R-a}{c}\right)$$

where the last series converges for $R>cq+a$ because of asymptotic formulas for the Legendre functions. Therefore, the stronger assertion

$$\sum_{n=0}^\infty\sum_{k=0}^\infty|T_{nk}|<\infty \tag{1.16}$$

is proved, where T_{nk} are the matrix coefficients of the operator T.

It can be shown that when condition (1.4) is satisfied the column of the right-hand sides of system (1.13) and (1.14) belongs to the space $l_1\subset l_2$. Then system (1.13) and (1.14) is correctly solvable for almost all values of the parameters therein in the Hilbert space l_2, and an approximate solution can be obtained by the method of reduction /5/. It follows from the solvability of the system in l_2 and (1.4), (1.13), (1.14) and (1.16) that

$$\sum_{k=0}^\infty|a_k^{(i)}|<\infty$$

which means absolute and uniform convergence of the series (1.5), (1.8) and (1.10).

Now, let stresses be given on the boundary (G is the shear modulus)

$$\begin{gathered}\mathbf{FU}|_{r=R}=2G\sum_{k=0}^\infty[A_{k,1}^{(1)}P_k^{(1)}(\cos\theta)\,e_\rho+A_{k,1}^{(2)}P_k(\cos\theta)\,e_z]\\ \mathbf{FU}|_{\xi_1=\xi_0}=2Ghc^{-1}\sum_{k=0}^\infty[A_{k,2}^{(1)}P_k^{(1)}(\cos\eta_1)\,e_\rho+A_{k,2}^{(2)}P_k(\cos\eta_1)\,e_z]\\ h=(q^2-\cos^2\eta_1)^{-1/2}\end{gathered} \tag{1.17}$$

The solution of the first fundamental problem will be sought, as before, in the form of (1.5). Going over from displacements to stresses on the appropriate boundary surfaces in (1.8) and (1.10) and satisfying conditions (1.17), we obtain an infinite system of linear algebraic Eqs.(1.11) with the following matrix coefficients

$$\begin{gathered}s_{k,1}^{(1)}=\frac{k}{k+1},\quad s_{k,2}^{(1)}=2\nu+\frac{(k-1)(k-2)}{2k-1},\quad s_{k,1}^{(2)}=k,\\ s_{k,2}^{(2)}=k\left[2\nu-1+\frac{k(k-2)}{2k-1}\right]\\ s_{k,1}^{(3)}=\frac{1}{Q_k(q)}\frac{d}{d\xi_0}Q_k^{-1}(q),\quad s_{k,2}^{(3)}=\frac{1}{Q_k(q)}\left[\frac{q^2}{k+1}\frac{d}{d\xi_0}\left(\frac{Q_{k+1}^{(1)}(q)}{q}\right)-2\nu Q_k(q)\right]\\ s_{k,1}^{(4)}=\frac{Q_k^{(1)}(q)}{Q_k(q)},\quad s_{k,2}^{(4)}=\frac{1}{Q_k(q)}\left[(k+1)\,q^2\frac{d}{d\xi_0}\left(\frac{Q_{k+1}(q)}{q}\right)+(1-2\nu)\,Q_k^{(1)}(q)\right]\\ t_{n,k}^{(1,1)}=\frac{(-1)^{k+n+1}c\,(k+1)!}{2kR^{k+1}Q_n(q)}C_{k,n}^{(1)},\quad t_{n,k}^{(2,1)}=\frac{(-1)^{k+n}c\,(k+1)!}{2R^{k+1}Q_n(q)}C_{k,n}^{(1)}\end{gathered} \tag{1.18}$$

$$t_{n,k}^{(1,2)} = \frac{(-1)^{k+n} ck!}{2R^{k+1}Q_n(q)} \left[\left(2\nu - \frac{(k+2)(k+3)}{2k+3}\right) C_{k,n}^{(1)} - \frac{k+1}{k} C_{k,n}^{(2)}\right]$$

$$t_{n,k}^{(2,2)} = \frac{(-1)^{k+n} c(k+1)!}{2R^{k+1}Q_n(q)} \left[C_{k,n}^{(2)} - \left(2\nu - 1 - \frac{(k+1)(k+3)}{2k+3}\right) C_{k,n}^{(1)}\right]$$

$$t_{n,k}^{(3,1)} = (-1)^{k+n+1}\left(k + \frac{1}{2}\right)\frac{n!}{R^n} \frac{d}{d\xi_0} P_k^{(-1)}(q) C_{n,k}^{(1)}$$

$$t_{n,k}^{(3,2)} = (-1)^{k+n}\left(k + \frac{1}{2}\right)\frac{n!}{R^n} \times$$

$$\left[\frac{d}{d\xi_0} P_k^{(-1)}(q) C_{n,k}^{(2)} - \left(\frac{q^2}{k}\frac{d}{d\xi_0}\left(\frac{P_{k-1}^{(1)}(q)}{q}\right) + 2\nu P_k(q)\right) C_{n,k}^{(1)}\right]$$

$$t_{n,k}^{(4,1)} = (-1)^{k+n+1}\left(k + \frac{1}{2}\right) P_k^{(1)}(q) \frac{n!}{R^n} C_{n,k}^{(1)}$$

$$t_{n,k}^{(4,2)} = (-1)^{k+n}\left(k + \frac{1}{2}\right)\frac{n!}{R^n} \times$$

$$\left[P_k^{(1)}(q) C_{n,k}^{(2)} - \left(kq^2 \frac{d}{d\xi_0}\left(\frac{P_{k-1}(q)}{q}\right) + (2\nu - 1) P_k^{(1)}(q)\right) C_{n,k}^{(1)}\right]$$

For $k = 0$ the equalities $a_0^{(1)} = a_0^{(2)} = a_0^{(3)} = 0$ must be appended to (1.11) for $i = 2$. The two mentioned equations have the form

$$a_0^{(4)} = A_{0,2}^{(2)} Q_0(q) \left[q^2 \frac{d}{d\xi_0}\left(\frac{Q_1(q)}{q}\right) + (1 - 2\nu) Q_0^{(1)}(q)\right]^{-1}, \quad a_0^{(4)} = -\frac{R^2 A_{0,1}^{(2)} Q_0(q)}{(2\nu - 2)c} \tag{1.19}$$

The statics conditions for this problem reduce to the relationship

$$c\bar{q} A_{0,2}^{(2)} = -R^2 A_{0,1}^{(2)} \tag{1.20}$$

from which it follows that one of Eqs.(1.19) is a corollary of the other, which means that the system is consistent for $k = 0$.

In the case of the first boundary-value problem the estimates

$$|\Delta_1| > 2\nu k(k+1)^{-1}, \quad |\Delta_2| > (1 - \nu) \operatorname{cth} \xi_0, \quad \nu < 1$$

are proved for the determinants Δ_1 and Δ_2.

Investigation of the solvability of the infinite system is completely analogous to that performed earlier for the second boundary-value problem.

2. Let us consider the first and second axisymmetric problems for (1.1) for a prolate spheroid with a spherical cavity whose centre is on the spheroid axis at a distance a from its centre. We introduce an identically directed spheroidal system of coordinates (ξ, η, φ) and a spherical system of coordinates (r_1, θ_1, φ) superposed on the centres of the boundary surfaces. Then relationships (1.2) must be replaced by

$$c \operatorname{ch} \xi \cos \eta = r_1 \cos \theta_1 + a, \quad c \operatorname{sh} \xi \sin \eta = r_1 \sin \theta_1 \tag{2.1}$$

Let the displacement vectors

$$\mathbf{U}|_{r_1=R} = \sum_{k=0}^{\infty} [A_{k,1}^{(1)} P_k^{(1)}(\cos\theta_1) e_\rho + A_{k,1}^{(2)} P_k(\cos\theta_1) e_z] \tag{2.2}$$

$$\mathbf{U}|_{\xi=\xi_0} = \sum_{k=0}^{\infty} [A_{k,2}^{(1)} P_k^{(1)}(\cos\eta) e_\rho + A_{k,2}^{(2)} P_k(\cos\eta) e_z]$$

subject to condition (1.4), be given on the boundary.

We will seek the solution of the problem in the form

$$\mathbf{U} = \sum_{k=1}^{2} \sum_{n=0}^{\infty} \left[a_n^{(k)} \frac{R^{n+1}}{n!} \mathbf{W}_{k,n}^{+}(r_1, \theta_1) + a_n^{(k+2)} \frac{\mathbf{U}_{k,n}^{-}(\xi, \eta)}{P_n(q)}\right] \tag{2.3}$$

where the notation taken in (1.6) is used.

Proceeding as in Sect.1, by using the expansion formulas of the external spherical solutions of the Lamé equation in external spheroidal solutions.

$$\mathbf{W}_{1,n}^{+}=\frac{2}{c}\sum_{k=n}^{\infty}\left(k+\frac{1}{2}\right)P_{k,n}^{(1)}\mathbf{U}_{1,k}^{+}$$

$$\mathbf{W}_{2,n}^{+}=\frac{2}{c}\sum_{k=n}^{\infty}\left(k+\frac{1}{2}\right)[P_{k,n}^{(1)}\mathbf{U}_{2,k}^{+}-P_{k,n}^{(2)}\mathbf{U}_{1,k}^{+}]$$

$$P_{n,k}^{(1)}=\frac{1}{c^k}\left[\frac{d^k}{dx^k}P_n(x)\right]_{|x=a/c},\quad P_{n,k}^{(2)}=\frac{R^2}{2k+3}P_{n,k+2}^{(1)}-cq^2P_{n-1,k+1}^{(1)}$$

and internal spheroidal solutions in internal spherical solutions /2/

$$\mathbf{U}_{1,n}^{-}=-\sum_{k=0}^{n}P_{n,k}^{(1)}\mathbf{W}_{1,k}^{-},\quad \mathbf{U}_{2,n}^{-}=\sum_{k=0}^{n}[P_{n,k}^{(1)}\mathbf{W}_{2,k}^{-}+P_{n,k}^{(2)}\mathbf{W}_{1,k}^{-}]$$

we represent the displacement (2.3) in spherical and spheroidal systems of coordinate and we satisfy the boundary conditions (2.2). The infinite system of linear algebraic equations has the form (1.11) with the following matrix coefficients

$$s_{k,1}^{(1)}=\frac{1}{k},\quad s_{k,2}^{(1)}=\frac{k+2}{2k+3},\quad s_{k,1}^{(2)}=-1,\quad s_{k,2}^{(2)}=4\nu-3-\frac{(k+1)^2}{2k+3},\tag{2.4}$$

$$s_{k,1}^{(4)}=-1$$

$$s_{k,1}^{(3)}=-\frac{P_k^{(-1)}(q)}{P_k(q)},\quad s_{k,2}^{(3)}=\frac{(k-1)qP_{k-1}^{(-1)}(q)}{P_k(q)},\quad s_{k,2}^{(4)}=\frac{(4\nu-3)P_k(q)+kqP_{k-1}(q)}{P_k(q)}$$

$$t_{n,k}^{(1,1)}=-\frac{R^kP_{n,k}^{(1)}}{(k+1)!\,P_n(q)},\quad t_{n,k}^{(2,1)}=-\frac{R^kP_{n,k}^{(1)}}{k!\,P_n(q)}$$

$$t_{n,k}^{(2,2)}=\frac{R^k}{k!\,P_n(q)}\left[P_{n,k}^{(2)}+\left(4\nu-3+\frac{k^2}{2k-1}\right)P_{n,k}^{(1)}\right]$$

$$t_{n,k}^{(3,1)}=-\frac{2k+1}{c}\frac{R^{n+1}}{n!}Q_k^{(-1)}(q)P_{k,n}^{(1)},\quad t_{n,k}^{(4,1)}=-\frac{2k+1}{c}\frac{R^{n+1}}{n!}Q_k(q)P_{k,n}^{(1)}$$

$$t_{n,k}^{(1,2)}=\frac{R^k}{k!P_n(q)}\left[\frac{P_{n,k}^{(2)}}{k+1}+\frac{k-1}{2k-1}P_{n,k}^{(1)}\right]$$

$$t_{n,k}^{(3,2)}=\frac{2k+1}{c}\frac{R^{n+1}}{n!}[Q_k^{-1}(q)P_{k,n}^{(2)}-(k+2)qQ_{k+1}^{(-1)}(q)P_{k,n}^{(1)}]$$

$$t_{n,k}^{(4,2)}=\frac{2k+1}{c}\frac{R^{n+1}}{n!}[Q_k(q)P_{k,n}^{(2)}+((4\nu-3)Q_k(q)-(k+1)qQ_{k+1}(q))P_{k,n}^{(1)}]$$

For $k=0$ equations $a_0^{(1)}=a_0^{(3)}=0$ must be added to (1.11) for $i=2$.

Lemma 2.1. For $k\geqslant 1$, $\nu<1/2$ the determinants Δ_1, Δ_2 of system (1.13) and (2.4) are non-zero for all $\xi_0>0$. The following estimates hold:

$$|\Delta_1|>(3-4\nu)(k+1)^{-1},\quad |\Delta_2|\geqslant(2-4\nu)(k+1)^{-1}\,\mathrm{th}\,\xi_0\tag{2.5}$$

Proof. We will prove the second estimate. We write Δ_2 in the explicit form

$$P_k(q)^2\Delta_2=\frac{3-4\nu}{k(k+1)}P_k^{(1)}(q)P_k(q)-\frac{q}{k+1}P_k^{(1)}(q)P_{k-1}(q)+\frac{q}{k}P_{k-1}^{(1)}(q)P_k(q)$$

It follows from the recursion formulas for the Legendre functions that

$$P_k^{(1)}(q)P_k(q)-kqP_k^{(1)}(q)P_{k-1}(q)+(k+1)qP_{k-1}^{(1)}(q)P_k(q)=$$
$$(2k+1)^{-1}[P_{k+1}^{(2)}(q)P_{k-1}^{(1)}(q)-P_{k+1}^{(1)}(q)P_{k-1}^{(2)}(q)]$$

The right-hand side of the last formula is expanded in a power series in $q-1$, which is strictly positive for $\xi_0>0$, $k\geqslant 1$. Taking account of the inequality $P_k^{(1)}(q)\geqslant k\,\mathrm{th}\,\xi_0 P_k(q)$, we obtain the required result.

Lemma 2.2. The operator T of system (1.13) with the matrix coefficients (2.4) is a completely continuous operator from l_2 into l_2 for $cq-a>R$ if $aq>c$ and for $\bar{q}\,(c^2-$

$a^2)^{1/2} > R$ if $aq \leqslant c$.

Proof. Consider an arbitrary matrix coefficient of the operator of the system T. It can have a finite sum of expressions of the form

$$\tau_{nk} = Bn^{\alpha} \frac{R^n}{n!c^n} Q_k^{(\gamma)}(q) \left| \frac{d^n}{dx^n} P_k(x) \right|_{x=a/c}$$

as upper bound, where B is a constant independent of n and k and α, γ are fixed non-negative integers.

Let us estimate the sum of the squares of τ_{nk}. To do this we consider the formula /6/

$$\frac{P_n^{(m)}(\cos\theta_1)}{r_1^{n+1}} = \frac{2}{c} \frac{(-1)^m}{(n-m)!} \sum_{k=n}^{\infty} \left(k+\frac{1}{2}\right) P_{k,n}^{(1)} Q_k^{(-m)}(\operatorname{ch}\xi) P_k^{(m)}(\cos\eta)$$

in which all the parameters are connected by the relationship (2.1). It is known that the functions

$$[(k+{}^1/_2)(k-m)!/(k+m)!]^{1/2} P_k^{(m)}(x)$$

form a complete orthonormal system in $L_2[-1,1]$. Let us write the Parseval equality for this system

$$\sum_{k=n}^{\infty}\left[\frac{2}{c(n-m)!}\left(\left(k+\frac{1}{2}\right)\frac{(k+m)!}{(k-m)!}\right)^{1/2} P_{k,n}^{(1)} Q_k^{(-m)}(q)\right]^2 = \int_0^{\pi}\left[\frac{P_n^{(m)}(\cos\theta_1)}{r_1^{n+1}}\right]^2 \sin\theta_1\, d\theta_1$$

Hence it follows that

$$\sum_{n=0}^{\infty} n^{2\alpha} R^{2n+2} \sum_{k=n}^{\infty}\left[\frac{2}{c(n-m)!}\left(\left(k+\frac{1}{2}\right)\frac{(k+m)!}{(k-m)!}\right)^{1/2} P_{k,n}^{(1)} Q_k^{(-m)}(q)\right]^2 =$$
$$\int_0^{\pi}\sum_{n=0}^{\infty} n^{2\alpha}\left(\frac{R}{r_1}\right)^{2n+2} [P_n^{(m)}(\cos\theta_1)]^2 \sin\theta_1\, d\theta_1.$$

The series on the right converges if $r_1 > R$. Finding the minimum r_1 we obtain that it equals $cq - a$ if $aq > c$ and $\bar{q}(c^2 - a^2)^{1/2}$ if $aq \leqslant c$. Therefore, on satisfying the conditions of the lemma, the series comprised of squares of τ_{nk} converges, from which the proof of the assertion follows.

Remark. The conditions imposed by Lemma 2.2 on the parameters of the problem have the following geometric meaning: they require that the boundary surfaces do not intersect.

The question of the solvability of the infinite system can be solved exactly as in Sect. 1.

We consider the first boundary-value problem for a spheroid with a spherical cavity. Let the stresses

$$\mathbf{F}\mathbf{U}_{|r_1=R} = 2G \sum_{k=0}^{\infty} [A_{k,1}^{(1)} P_k^{(1)}(\cos\theta_1) e_\rho + A_{k,1}^{(2)} P_k(\cos\theta_1) \mathbf{e}_z] \tag{2.6}$$

$$\mathbf{F}\mathbf{U}_{|\xi=\xi_0} = 2Ghc^{-1} \sum_{k=0}^{\infty} [A_{k,2}^{(1)} P_k^{(1)}(\cos\eta) e_\rho + A_{k,2}^{(2)} P_k(\cos\eta) e_z]$$

be given on the boundary.

We will seek the solution of the problem in the form (2.3). Carrying out the above-mentioned calculations and changing from displacements to stresses, after satisfying the boundary conditions we obtain an infinite system of linear algebraic Eqs.(1.11) with the following matrix coefficients

$$\begin{gathered}
s_{k,1}^{(1)} = \frac{k+1}{kR}, \quad s_{k,2}^{(1)} = \frac{1}{R}\left[\frac{(k+2)(k+3)}{2k+3} - 2\nu\right] \\
s_{k,1}^{(2)} = -\frac{k+1}{R}, \quad s_{k,2}^{(2)} = \frac{k+1}{R}\left[2\nu - 1 - \frac{(k+1)(k+3)}{2k+3}\right] \\
s_{k,1}^{(3)} = -\frac{1}{P_k(q)} \frac{d}{d\xi_0} P_k^{(-1)}(q), \quad s_{k,2}^{(3)} = \frac{1}{P_k(q)}\left[\frac{q^2}{k}\frac{d}{d\xi_0}\left(\frac{P_{k-1}^{(1)}(q)}{q}\right) + 2\nu P_k(q)\right] \\
s_{k,1}^{(4)} = -\frac{P_k^{(1)}(q)}{P_k(q)}, \quad s_{k,2}^{(4)} = \frac{1}{P_k(q)}\left[kq^2 \frac{d}{d\xi_0}\left(\frac{P_{k-1}(q)}{q}\right) + (2\nu - 1) P_k^{(1)}(q)\right]
\end{gathered} \tag{2.7}$$

$$t_{n,k}^{(1,1)} = \frac{kR^{k-1}}{(k+1)!} \frac{P_{n,k}^{(1)}}{P_n(q)}, \quad t_{n,k}^{(2,1)} = \frac{R^{k-1}}{(k-1)!} \frac{P_{n,k}^{(1)}}{P_n(q)}$$

$$t_{n,k}^{(1,2)} = -\frac{R^{k-1}}{k!}\left[\frac{k}{k+1}\frac{P_{n,k}^{(2)}}{P_n(q)} + \left(2\nu + \frac{(k-1)(k-2)}{2k-1}\right)\frac{P_{n,k}^{(1)}}{P_n(q)}\right]$$

$$t_{n,k}^{(2,2)} = -\frac{R^{k-1}}{(k-1)!}\left[\frac{P_{n,k}^{(2)}}{P_n(q)} + \left(2\nu - 1 + \frac{k(k-2)}{2k-1}\right)\frac{P_{n,k}^{(1)}}{P_n(q)}\right]$$

$$t_{n,k}^{(3,1)} = -\frac{2k+1}{c}\frac{R^{n+1}}{n!}\frac{d}{d\xi_0}Q_k^{(-1)}(q)\,P_{k,n}^{(1)}, \quad t_{n,k}^{(4,1)} = -\frac{2k+1}{c}\frac{R^{n+1}}{n!}Q_k^{(1)}(q)\,P_{k,n}^{(1)}$$

$$t_{n,k}^{(3,2)} =$$

$$\frac{2k+1}{c}\frac{R^{n+1}}{n!}\left[\frac{d}{d\xi_0}Q_k^{(-1)}(q)\,P_{k,n}^{(2)} - \left(\frac{q^2}{k+1}\frac{d}{d\xi_0}\left(\frac{Q_{k+1}^{(1)}(q)}{q}\right) - 2\nu Q_k(q)\right)P_{k,n}^{(1)}\right]$$

$$t_{n,k}^{(4,2)} = \frac{2k+1}{c}\frac{R^{n+1}}{n!}\times$$

$$\left[Q_k^{(1)}(q)\,P_{k,n}^{(2)} - \left((k+1)\,q^2\frac{d}{d\xi_0}\left(\frac{Q_{k+1}(q)}{q}\right) + (1-2\nu)\,Q_k^{(1)}(q)\right)P_{k,n}^{(1)}\right]$$

For $k = 0$ the equalities $a_0{}^{(1)} = a_0{}^{(3)} = a_0{}^{(4)} = 0$ must be added to (1.11) for $i = 2$. These equations have the form

$$\frac{2\nu-2}{R}a_0^{(2)} = A_{0,1}^{(2)}, \quad -\frac{R}{c}\left[q^2\frac{d}{d\xi_0}\left(\frac{Q_1(q)}{q}\right) + (1-2\nu)\,Q_0^{(1)}(q)\right]a_0^{(2)} = A_{0,2}^{(2)} \tag{2.8}$$

It follows from the statics conditions (1.20) that one of the Eqs.(2.8) is a corollary of the other, i.e., the system is consistent for $k = 0$.

In the case of the first boundary-value problem the following estimates are obtained for the determinants Δ_1 and Δ_2:

$$|\Delta_1| > (k+1)R^{-2}, \quad |\Delta_2| > k(k+1)^{-1}2\nu \operatorname{th}\xi_0$$

The solvability of the infinite system is established as before.

3. As an application we will consider the problem of an external hydrostatic pressure p acting on a sphere with a centrally located force-free spheroidal cavity. The conditions on the boundary have the form (1.17) where

$$A_{k,1}^{(1)} = \frac{p}{2G}\delta_{k1}, \quad A_{k,1}^{(2)} = -\frac{p}{2G}\delta_{k1}, \quad A_{k,2}^{(1)} = A_{k,2}^{(2)} = 0; \quad \delta_{k1} = \begin{cases} 1, & k = 1 \\ 0, & k \neq 1 \end{cases}$$

The solution of the problem is given by (1.5), where $a_0{}^{(i)} = 0$, $a_k{}^{(i)}\ (i = 1, \ldots, 4)$ satisfy the infinite system of linear algebraic Eqs.(1.13) with the matrix coefficients (1.18) evaluated at $a = 0$. We will seek the solution of system (1.13) in the form of series in the small parameter $\varepsilon = cq/R$. Expanding the unknown $a_k{}^{(i)}$ and the matrix coefficients in a power series in ε, substituting them into (1.13) and equating coefficients of identical powers of ε we have

$$a_1^{(1)} = \frac{p}{2G}\left\{\frac{2-4\nu}{1+\nu} + \frac{2^3\varepsilon^3}{3q^2\Delta_1\Delta_2 Q_1(q)^2}\left[-\left((1-2\nu)\,3q^2 + \frac{4-\nu}{5} -\right.\right.\right.$$

$$\left.\nu(3-5\nu)\right)Q_1^{(1)}(q) -$$

$$\left.\left.\left((1-2\nu)\,6q^2 + \frac{2}{5}(4-\nu) - 2(1-\nu-\nu^2)\right)\frac{\bar{q}}{q}Q_1(q)\right]\right\} + O(\varepsilon^6)$$

$$a_1^{(2)} = \frac{p}{2G}\left\{\frac{3}{2+2\nu} + \frac{2\varepsilon^3}{3q^2\Delta_1\Delta_2 Q_1(q)^2}\left[-\left(\frac{9}{4}q^2 + \frac{1}{4} - 2\nu\right)Q_1^{(1)}(q) -\right.\right.$$

$$\left.\left(\frac{9}{2}q^2 - 1 - \nu\right)\frac{\bar{q}}{q}Q_1(q)\right\} + O(\varepsilon^6)$$

$$a_1^{(3)} = \frac{p}{2G}\frac{\varepsilon}{\Delta_2 Q_1(q)}\times$$

$$\left[\left(\frac{3}{2}q^2 + 1 - 2\nu\right)Q_1^{(1)}(q) + (3q^2 + 1 - 2\nu)\frac{\bar{q}}{q}Q_1(q)\right] + O(\varepsilon^4)$$

$$a_1^{(4)} = \frac{p}{2G}\frac{\varepsilon}{\Delta_2 Q_1(q)}\left[-\frac{1}{2}Q_1^{(1)}(q) - \frac{\bar{q}}{q}Q(q)\right] + O(\varepsilon^4)$$

$$a_k{}^{(i)} = o(\varepsilon^3), \ k = 2, 3, \ldots; \ i = 1, \ldots, 4$$

Then we obtain for the stress on the sphere surface for $\theta = 0$

$$\sigma_\theta = \sigma_\varphi = p\left\{-1 + \frac{\varepsilon^3}{q^2\Delta_2 Q_1(q)^2}\left[-\left(\frac{3}{2}q^2 + \frac{1}{2} - \nu\right)Q_1^{(1)}(q) - 3q\bar{q}Q_1(q)\right]\right\} + O(\varepsilon^4) \qquad (3.1)$$

The first component in (3.1) corresponds to the case of a homogeneous sphere and agrees with the known values of σ_θ and σ_φ in the Lamé problem. The first correction to it is of the third order of smallness and, as can be shown, is negative for $\nu < {}^1/_2$. Therefore, the presence of a spheroidal cavity results in an increase in the compressive stresses σ_θ and σ_φ on the sphere surface for $\theta = 0$. If we let $q \to \infty$ in (3.1) for $c = R_1/q$, then the spheroidal cavity becomes a spherical cavity and the stress equals

$$\sigma_\theta = \sigma_\varphi = p\left[-1 - {}^3/_2\zeta^3\right] + o(\zeta^3), \quad \zeta = R_1/R$$

The latter formula agrees with the two first terms of the expansions of $\sigma_\theta, \sigma_\varphi$ in power series in ζ in the Lamé problem of determining the state of stress of a hollow sphere subjected to uniform external pressure /7/.

REFERENCES

1. PROTSENKO V.S. and NIKOLAYEV A.G., The solution of spatial problems of elasticity theory by using re-expansion formulas, Prikl. Mekhan., 22, 7, 1986.
2. NIKOLAYEV A.G., Re-expansion formulas for vector solutions of the Lamé equation in spherical and spheroidal coordinate systems. Mathematical Methods of Analysing Dynamic Systems, 8, Khar'kov Aviation Institute, 1984.
3. PODIL'CHUK YU.N., Three-dimensional Problems of Elasticity Theory. Naukova Dumka, Kiev, 1979.
4. ULITKO A.F., The Eigenvector Function Method in Spatial Problems of Elasticity Theory. Naukova Dumka, Kiev, 1979.
5. KANTOROVICH L.V. and AKILOV G.P., Functional Analysis. Nauka, Moscow, 1977.
6. PROTSENKO V.S. and NIKOLAYEV A.G., Re-expansion formulas for solutions of the Laplace equation in spherical and prolate spheroidal coordinates. Dokl. Akad. Nauk UkrSSR, Ser. A., 12, 1983.
7. LUR'YE A.I., Spatial Problems of Elasticity Theory. Gostekhizdat, Moscow, 1955.

Translated by M.D.F.